Topology and its Applications 156 (2009) 2123–2128

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Completion of quasi-topological groups

Barbora Batíková

University of Economics, Department of Mathematics, Ekonomická 957, 148 00 Prague 4, Czech Republic

A R T I C L E I N F O

MSC:
54E15

Keywords:
Completion
Quasi-topological group
Cauchy filter
Epireflective

A B S T R A C T

A topology of a quasi-topological group is induced by several natural semi-uniformities, namely right, left, two-sided and Roelcke semi-uniformities. A quasi-topological group is called complete if every Cauchy (in some sense—we examine several generalizations of Cauchy properties) filter on the two-sided semi-uniformity converges.

We use the theory of Hausdorff complete semi-uniform spaces, see [B. Batíková, Completion of semi-uniform spaces, Appl. Categor. Struct. 15 (2007) 483–491], and show that Hausdorff complete quasi-topological groups form an epireflective subcategory of Hausdorff quasi-topological groups. But the reflection arrows need not be embeddings.

For several types of Cauchy-like properties we show examples of quasi-topological groups that cannot be embedded into a complete group.

The theme of this paper is to study completions of quasi-topological groups. Quasi-topological groups are groups with topology where multiplication is separately continuous (not necessarily continuous) and inversion is continuous. Another generalization of topological groups are paratopological groups. See the survey paper [7] for some results and for other references.

In the present paper we show several generalizations of Cauchy properties to quasi-topological groups such that the corresponding completeness is epireflective in Hausdorff quasi-topological groups. But one never gets embeddings of all groups into their group completions.

For several generalizations of Cauchy properties we show examples of T_1- and T_2-quasi-topological groups that cannot be embedded into a complete quasi-topological group.

1. Notations and definitions

The main references are the books [4,5] for topology and [2,8] for topological structures on groups.

On topological groups there are several natural uniformities inducing the original group topology, namely right, left, upper (two-sided) and lower (Roelcke) uniformities. The situation is similar if we demand only separate continuity of the multiplication mapping and instead of uniformities consider semi-uniformities. First recall some basic definitions of topological structures on groups:

Definition 1. Let X be a group with a topology τ.

We say that X is *a right topological group* if the mapping $m : X \times X \to X : (x, y) \mapsto xy$ is continuous for the left variable, which means that for each $y \in X$ the mapping (*the right translation*) $r_y : x \mapsto xy$ is continuous.

We say that X is *a left topological group* if the mapping m is continuous for the right variable, which means that for each $x \in X$ the mapping (*the left translation*) $l_x : y \mapsto xy$ is continuous.

E-mail address: stepkova@vse.cz.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.03.035

We say that X is *a semi-topological group* if the mapping m is separately continuous, which means that it is continuous for the right and left variable.

We say that X is *a quasi-topological group* if it is a semi-topological group and the inversion mapping $i : X \to X : x \mapsto x^{-1}$ is continuous.

A group X with a topology is a quasi-topological group iff all the right, resp. left, translations are homeomorphisms. Any right, resp. left, topological group with a continuous inversion map is a quasi-topological group.

We recall from [4] that a semi-uniformity on a set X is a filter of subsets of $X \times X$ having a base of symmetric sets and the intersection of which contains the diagonal. Every semi-uniformity $\mathscr{U}$ on X induces a closure on X, namely $\overline{A} = \bigcap_{U \in \mathscr{U}} U[A]$. We shall deal with those semi-uniformities inducing topological closures. They can be described as follows:

Definition 2. A semi-uniformity $\mathscr{U}$ on X is called *a t-semi-uniformity* if

$$(\forall U \in \mathscr{U})(\forall x \in X)(\exists V \in \mathscr{U})\big(\forall y \in V[x]\big)\big(U[x] \in \mathscr{U}_y\big),$$

where $\mathscr{U}_y = \{U[y];\ U \in \mathscr{U}\}$.

The couple $(X, \mathscr{U})$ is then called *a t-semi-uniform space*, where the letter t stands for *topological.*

We describe several semi-uniformities on quasi-topological groups, the description is the same as for topological groups, see e.g. [8]:

Definition 3. The semi-uniformity $\mathscr{R}$ with the base consisting of sets $R_U = \{(x, y) \in X \times X;\ \ yx^{-1} \in U\}$, $U \in \mathscr{U}_e$, where $\mathscr{U}_e$ is a neighbourhood filter of the unit e in the topology of a quasi-topological group X, is called the *right semi-uniformity* on the group X.

The semi-uniformity $\mathscr{L}$ with the base consisting of sets $L_U = \{(x, y) \in X \times X;\ \ x^{-1}y \in U\}$, $U \in \mathscr{U}_e$, is called the *left semi-uniformity* on the group X.

Supremum $\mathscr{L} \vee \mathscr{R}$, resp. infimum $\mathscr{L} \wedge \mathscr{R}$, is clearly also a semi-uniformity on X, see e.g. [4], and it is called the *two-sided* (or *upper*), resp. *Roelcke* (or *lower*), *semi-uniformity* on X.

Remark that each of the described semi-uniformities is *topological* (or t-) *semi-uniformity*. They have one more useful property:

Definition 4. A t-semi-uniformity on a set X is called *pointwise open* if it has such a base $\mathscr{B}$ that $B[x]$ is an open neighborhood of x, for all $x \in X$, $B \in \mathscr{B}$.

In the next we examine only the two-sided semi-uniformities, which are used in the definition of complete quasi-topological groups.

2. Initial t-semi-uniformities on groups

We need some bacis facts about quasi-topological groups, which can be proved in the same way as in topological groups, see e.g. [8].

Proposition 1. *Let X, Y be quasi-topological groups, $f : X \to Y$ a continuous homomorphism. Then f is uniformly continuous with respect to the right, left, resp. to the two-sided, resp. to the lower, semi-uniformities on X and Y.*

Proposition 2. *Let (X_i, τ_i) be a quasi-topological group for each $i \in I$. Let X be a group and $f_i : X \to X_i$ be a homomorphism, for each $i \in I$, τ the initial topology on X w.r.t. the maps $f_i : X \to (X_i, \tau_i)$. Then (X, τ) is a quasi-topological group. Its right semi-uniformity coincides with the initial semi-uniformity w.r.t the maps $f_i : X \to (X_i, \mathscr{R}_i)$, where $\mathscr{R}_i$ is the right semi-uniformity on (X_i, τ_i), $i \in I$. The same holds for the left and two-sided semi-uniformities.*

Proposition 3.

(a) *The subgroup of a quasi-topological group with the subspace topology is a quasi-topological group.*
(b) *Let (X, τ_i) be quasi-topological groups, $i \in I$. The group (X, τ) with the supremum topology $\tau = \sup_{i \in I} \tau_i$ is a quasi-topological group.*
(c) *Let (X, τ_i) be quasi-topological groups, $i \in I$. The topological product $(\prod_{i \in I} X_i, \prod_{i \in I} \tau_i)$ is a quasi-topological group.*

Proposition 4. *A closure of a subgroup X of a quasi-topological group Y is a quasi-topological subgroup of Y.*

3. Categories of Hausdorff complete groups

The definitions of Cauchy-like properties and completeness come from [3]:

Definition 5. Let $(X, \mathscr{U})$ be a t-semi-uniform space.

(1) A filter $\mathfrak{f}$ on a space X is called *a classic Cauchy filter on* $(X, \mathscr{U})$ if for every $U \in \mathscr{U}$ there exists $F \in \mathfrak{f}$ such that $F \times F \subset U$.
(2) A filter $\mathfrak{f}$ on a space X is called *a Cauchy filter on* $(X, \mathscr{U})$ if for every $U \in \mathscr{U}$ there exists $F \in \mathfrak{f}$ such that $U[x] \in \mathfrak{f}$ for every $x \in F$.
(3) A filter $\mathfrak{f}$ on a space X is called *a weak Cauchy filter on* $(X, \mathscr{U})$ if for every $U \in \mathscr{U}$ and every $F \in \mathfrak{f}$ there exists $x \in F$ such that $U[x] \in \mathfrak{f}$.
(4) A filter $\mathfrak{f}$ on a space X is called *a semi-Cauchy filter on* $(X, \mathscr{U})$ if for every $U \in \mathscr{U}$ there exists $x \in X$ such that $U[x] \in \mathfrak{f}$.

Each of the previous properties implies the next one in t-semi-uniform spaces, but not conversely. In uniform spaces they are equivalent.

Definition 6. A t-semi-uniform space is called (*classic*, *weak*, *semi*-) *complete* if every (classic, weak, semi-) Cauchy filter converges.

Every (Hausdorff) topological group is embedded in a (Hausdorff) topological group that is complete in its two-sided uniformity. The Hausdorff complete (in the two-sided uniformity) groups form an epireflective subcategory of the category of all Hausdorff topological groups with continuous homomorphisms. We will examine the situation in quasi-topological groups.

Definition 7. A quasi-topological group is called (*classic*, *weak*, *semi*-) *complete* if it is (classic, weak, semi-) complete in its two-sided semi-uniformity.

Notation. Denote by $(\mathtt{cl}, \mathtt{s}, \mathtt{w})\mathtt{Compl}_{\mathtt{Grp}}$ the category of all (classic, semi-, weak) complete Hausdorff quasi-topological groups with continuous homomorphisms, $\mathtt{QGrp}$ the category of all Hausdorff quasi-topological groups with continuous homomorphisms.

Theorem 1. *The categories* $\mathtt{clCompl}_{\mathtt{Grp}}$, $\mathtt{sCompl}_{\mathtt{Grp}}$, $\mathtt{wCompl}_{\mathtt{Grp}}$, $\mathtt{Compl}_{\mathtt{Grp}}$ *are epireflective subcategories of* $\mathtt{QGrp}$.

Proof. It follows from the fact that Hausdorff (classic, semi-, weak) complete spaces are closed under products and closed subspaces, see [3], and Propositions 3 and 4. If the subcategory of $\mathtt{QGrp}$ is closed under products and closed subgroups then it is epireflective. □

4. Completion of quasi-topological groups

We showed that complete (for all of our definitions) quasi-topological groups form an epireflective subcategory of quasi-topological groups. In the category of Hausdorff topological groups the reflection arrows (to Hausdorff complete reflections) are embeddings. In all the categories $(\mathtt{cl}, \mathtt{s}, \mathtt{w})\mathtt{Compl}_{\mathtt{Grp}}$ the situation is different.

Definition 8. Let X be a quasi-topological group. A (classic, weak, semi-) complete quasi-topological group that contains X as a dense subgroup is *a group* (*classic*, *weak*, *semi*-) *completion* of the group X.

We show that not every Hausdorff quasi-topological group has a Hausdorff group classic completion. That means that the reflection arrows of Hasdorff quasi-topological groups into Hausdorff (classic, weak, semi-) complete quasi-topological groups need not be embeddings:

Example 1. Take the Abelian group $\mathbb{R} \times \mathbb{R}$. Denote $e = (0; 0)$ and $a_n = (\frac{1}{n}, \sin\frac{1}{n}) \in \mathbb{R} \times \mathbb{R}$, $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers. Denote $B_\varepsilon = \{x \in \mathbb{R} \times \mathbb{R};\ d(x, e) < \varepsilon\}$, where d is the Euclidean metric on $\mathbb{R} \times \mathbb{R}$. For all points $x \in \mathbb{R} \times \mathbb{R}$ define sets $\mathscr{B}_x$:

for $x = e \in \mathbb{R} \times \mathbb{R}$ let $\mathscr{B}_x = \{B_\varepsilon \setminus \{a_n, -a_n\}_{n \in \mathbb{N}}\}_{\varepsilon > 0}$,
for $x \in \mathbb{R} \times \mathbb{R}$ let $\mathscr{B}_x = \{G_e + x\}_{G_e \in \mathscr{B}_e}$.

First we show that the system $\{\mathscr{B}_x\}_{x \in X}$ generates a topology on $\mathbb{R} \times \mathbb{R}$. It suffices to show that if $y \in G_x \in \mathscr{B}_x$ then there is a $G_y \in \mathscr{B}_y$ that $G_y \subset G_x$, see Proposition 1.2.3 in [5]. In fact, if $y \in G_x = G \setminus \{a_n + x, -a_n + x\}_{n \in \mathbb{N}}$, $y \neq x$, where G is a standard open neighborhood of x in $\mathbb{R} \times \mathbb{R}$, then there is a standard open neighborhood H of y that $H \subset G$. As $\{a_n + x\}_{n \in \mathbb{N}}$

and $\{-a_n + x\}_{n\in\mathbb{N}}$ converge to x in the standard topology, which is Hausdorff, we can assume that H contains only finitely many $a_n + x, -a_n + x$, and thus we can take H without points $a_n + x, -a_n + x$. Now $H \subset G_x$.

Take $X = \mathbb{R} \times \mathbb{R}$ with the topology τ generated by the neighborhood system $\{\mathscr{B}_x\}_{x\in X}$. Clearly, it is a Hausdorff quasi-topological group.

Take the filter $\mathfrak{f}$ generated by the set $\{\{a_n;\ n \geqslant n_0\},\ n_0 \in \mathbb{N}\}$.

We show that $\mathfrak{f}$ is classic Cauchy on X (with the right (denoted by $\mathscr{R}$), left or two-sided semi-uniformities, which are the same on Abelian groups).

Take $U \in \mathscr{B}_e$ and $R_U = \{(x, y) \in X \times X;\ x - y \in U\} \in \mathscr{R}$. We find a set $F \in \mathfrak{f}$ that $F \times F \subset R_U$, which means that we find a number $n_0 \in \mathbb{N}$ that $(a_n, a_m) \in R_U$ for all $n, m \geqslant n_0$. As $U = G \setminus \{a_n, -a_n\}_{n\in\mathbb{N}}$ for some standard neighborhood G of e and the sequence $\{a_n,\ n \in \mathbb{N}\}$ converges to e, and thus it is Cauchy, in the standard topology, there is an $n_0 \in \mathbb{N}$ that $a_n - a_m \in G$ for all $n, m \geqslant n_0$.

There is no $p \in \mathbb{N}$ that $a_n - a_m = a_p$. In fact, it would mean that there is a $p \in \mathbb{N}$ that $\frac{1}{n} - \frac{1}{m} = \frac{1}{p}$ and $\sin\frac{1}{n} - \sin\frac{1}{m} = \sin\frac{1}{p}$, and thus $\sin\frac{1}{n} - \sin\frac{1}{m} = \sin(\frac{1}{n} - \frac{1}{m}) = \sin\frac{1}{n}\cos\frac{1}{m} - \sin\frac{1}{m}\cos\frac{1}{n}$, which is impossible because the function $h(x) = \frac{\sin x}{1-\cos x}$ is injective in the interval $]0; \frac{\pi}{2}[$. Thus $a_n - a_m \in U$ for all $n, m \geqslant n_0$, and $\mathfrak{f}$ is classic Cauchy, and, of course, it has no limit in X.

If there is a Hausdorff classic completion Y of the space X the filter $\mathfrak{f}$ is classic Cauchy on Y, see [3, Proposition 1], and it converges to a point $y \in Y$. We show that e and y cannot be separated. Let there be disjoint open sets $G, H \subset Y$ such that $e \in G$, $y \in H$. First remind that every neighborhood G of e intersects every neighborhood of every point a_n, $n \geqslant n_0$ for some $n_0 \in \mathbb{N}$. As $\mathfrak{f}$ converges to y, and thus the sequence $\{a_n,\ n \in \mathbb{N}\}$ also converges to y, we can take such n_0 that $a_n \in H$ for all $n \geqslant n_0$, and thus H is a neighborhood of all a_n, $n \geqslant n_0$. That is a contradiction.

The quasi-topological group X has no (classic, weak, semi-) completion containing X as a subgroup.

In [3, Theorem 5], a completion (weak reflection) is constructed for all non-complete semi-uniform spaces. But this completion cannot be considered a quasi-topological group, because it contains the original space as an open dense subspace.

We show an easy example of T_1-quasi-topological group that has not a weak (and thus not a semi-) completion (in t-semi-uniform spaces), and consequently it has not a group weak (and thus not a semi-) completion:

Example 2. Let X be an Abelian quasi-topological group of integers with the coarse T_1-topology.

The filter $\mathfrak{f} = \{F \subset X;\ |X \setminus F| < \omega \wedge \{a, b\} \subset F\}$ is a weak Cauchy filter on the two-sided semi-uniformity on X, for every couple $a, b \in X$, $a \neq b$.

In fact, let U be an arbitrary neighbourhood of the neutral element 0, $F \in \mathfrak{f}$.

The sets $a - U, b - U, F$ have finite complements, and, thus, a non-empty intersection. Take an element z from this intersection. Now $z \in F$ and $a, b \in z + U$. Then $U[z] = z + U \in \mathfrak{f}$, and $\mathfrak{f}$ is weak Cauchy.

If the space X has a weak completion Y, then the filter $\mathfrak{f}$ is a weak Cauchy filter on Y, see [3, Proposition 1]. As Y is weak complete, the filter $\mathfrak{f}$ converges to a point $y \in Y$. This means that every neighborhood of y belongs to the filter $\mathfrak{f}$, and thus it contains the points a, b. Now $y \in \overline{\{a\}}^Y$, which is equivalent to $a \in \overline{\{y\}}^Y$, because the space Y is symmetric. From $a \in \overline{\{y\}}^Y$ and $y \in \overline{\{b\}}^Y$ it follows that $a \in \overline{\{b\}}^Y$. This is a contradiction to the topology of X.

The next useful Lemma does not hold generally in semi-uniformities:

Lemma 1. *In pointwise open t-semi-uniform spaces convergent filters are Cauchy.*

Proof. Let $(X, \mathscr{U})$ be a pointwise open t-semi-uniform space, a filter $\mathfrak{f}$ converge to a point $x \in X$. Take $B \in \mathscr{B}$ from Definition 4 and $U = U^{-1} \in \mathscr{U}$ such that $U \subset B$. If $y \in F = U[x] \in \mathfrak{f}$ then $x \in U[y] \subset B[y]$ and $B[y] \in \mathfrak{f}$ because it is a neighborhood of all its points. Thus $\mathfrak{f}$ is Cauchy. □

We can use the following proposition for proving the absence of a group weak or semi-completion:

Proposition 5. *If a pointwise open t-semi-uniform space X has a pointwise open semi-completion, resp. a pointwise open weak completion, then every semi-Cauchy, resp. weak Cauchy, filter on X must be Cauchy on X.*

Proof. Let Y be a pointwise open semi-completion of X, $\mathfrak{f}$ be a semi-Cauchy filter on X. Then $\mathfrak{f}$ generates a filter $\mathfrak{f}'$ that is semi-Cauchy on Y (see [3, Proposition 1]) and thus convergent on Y. Then by Lemma 1 $\mathfrak{f}'$ is Cauchy on Y, and thus Cauchy on X.

The proof for the weak completion is the same. □

In the next example we show once again that the T_1-quasi-topological group from Example 2 has not a group weak (and thus not semi-) completion. In fact, if it had a group weak (semi-) completion, weak (semi-) Cauchy filters would coincide with Cauchy ones, see Proposition 5.

Example 3. Let X be an Abelian quasi-topological group of integers with the cofinite topology. The filter $\mathfrak{f} = \{F \subset X;\ |X \setminus F| < \omega \wedge \{a,b\} \subset F\}$ is a weak Cauchy but not a Cauchy filter in the two-sided semi-uniformity on X for every couple $a, b \in X$, $a \neq b$.

In Example 2 we showed that $\mathfrak{f}$ is weak Cauchy.

The filter $\mathfrak{f}$ is not Cauchy. That would mean that it converges to any point in $\bigcap \mathfrak{f} = \{a, b\}$.

The next examples show Hausdorff quasi-topological groups that do not have group weak (and thus semi-) completion. We will use Proposition 5 as in Example 3.

First recall examples of quasi-topological groups described in [1] and [6]:

In [1, par. 3], some orbits are shown to be quasi-topological groups:

Example 4. Let $(G, +)$ be an Abelian group, Z a Hausdorff topological space. If $\phi : G \times Z \to Z$ is an action (i.e., $\phi(0, z) = z$, $\phi(a, \phi(b, z)) = \phi(a+b, z)$ for every $a, b \in G, z \in Z$, and $\phi(a, -) : Z \to Z$ is a continuous mapping for every $a \in G$), then every orbit $(O(z_0) = \{\phi(a, z_0);\ a \in G\} \subset Z)$ forms a semi-topological group. If G is an Abelian group of order 2, the orbits will be quasi-topological groups.

A special orbit is described in [6, Theorem 4]:

Example 5 *(Korovin's orbit).* Let $(G, +)$ be an Abelian group, X a Hausdorff topological space such that $|G^\omega| = |G| \geqslant |X| \cdot \omega$. In Example 4 let $Z = X^G$, ϕ be the shift $\phi(a, f)(b) = f(a+b)$.

Denote by $\mathscr{A}$ the set of all finite subsets of G, $\mathscr{F}(A)$ the set of all mappings of A to X for $A \in \mathscr{A}$, $\mathscr{F} = \bigcup\{\mathscr{F}(A);\ A \in \mathscr{A}\}$. Denote by $A(f)$ such a set from $\mathscr{A}$ that $f \in \mathscr{F}(A(f))$, for $f \in \mathscr{F}$.

Denote $\tau = |G| = |\mathscr{A}| = |\mathscr{F}|$.

Take $\{f_\alpha;\ \alpha \in \tau\}$ a well-ordering of the set $\mathscr{F}$.

By transfinite induction we pick such a subset $\{g_\alpha\colon \alpha \in \tau\}$ of G that the family $\{g_\alpha + A(f_\alpha)\colon \alpha \in \tau\}$ is disjoint.

Suppose that for an ordinal $\beta \in \tau$ we have a subset $\{g_\alpha\colon \alpha \in \beta\}$ of G such that the family $\{g_\alpha + A(f_\alpha)\colon \alpha \in \beta\}$ is disjoint.

Take an element $g_\beta \in G \setminus H$, where H is a minimal subgroup of G containing the set $\{A(f_\alpha)\colon \alpha \leqslant \beta\} \cup \{g_\alpha\colon \alpha < \beta\}$. $G \setminus H \neq \emptyset$ because $|H| \leqslant |\beta| \cdot \omega < \tau = |G|$.

Now the family $\{g_\alpha + A(f_\alpha)\colon \alpha \leqslant \beta\}$ is disjoint.

Then take a mapping $f \in X^G$ such that $f|_{g_\alpha + A(f_\alpha)} = f_\alpha \circ l_{g_\alpha}^{-1}|_{g_\alpha + A(f_\alpha)}$ for any $\alpha \in \tau$.

The subspace $Y = \{f_g = f \circ l_g;\ g \in G\}$ of the space X^G is a semi-topological group, its elements are functions f_g, $g \in G$, where $f_g(a) = f(g+a)$, for $a \in G$, with the group operation $f_g + f_h = f_{g+h}$. The neutral element is $f_0 = f$, inverse elements are $-f_g = f_{-g}$.

The subbasic neighbourhoods of the neutral element $f = f_0$ are sets $U_c^f = \{f_g;\ f_g(c) = f(g+c) \in U\}$, where U are neighbourhoods of $f(c)$ in X. The subbasic neighbourhoods of an element f_g are sets $U_c^{f_g} = \{f_h;\ f_h(c) = f(h+c) \in U\} = \{f_{g+k};\ f_{g+k}(c) = f(g+k+c) = f_k(g+c) \in U\} = f_g + U_{g+c}^f$, where U are neighbourhoods of $f_g(c) = f(g+c)$ in X.

If we take for G a group of order 2 then Y is a quasi-topological group. (Here $l_g^{-1} = l_g$.)

Lemma 2. *Take G, X, f from Example* 5. *For any finite sets $\{x_0, \ldots, x_m\} \subset X$ and $\{d_0, \ldots, d_m\} \subset G$, where $d_i \neq d_j$ for $i \neq j$, $i, j = 0, \ldots, m$, there is a $g_\alpha \in G$ (chosen by the transfinite induction in Example* 5) *such that $f_{g_\alpha}(d_i) = f(g_\alpha + d_i) = x_i$, $i = 0, 1, \ldots, m$. Moreover $g_\alpha + d_i \neq 0$ for all $i = 0, \ldots, m$.*

Proof. For the finite mapping h assigning x_i to d_i, $i = 0, \ldots, m$, there must exist an $\alpha \in \tau$ that $h = f_\alpha \in \mathscr{F}$ and $A(f_\alpha) = \{d_0, \ldots, d_m\}$. From $f|_{g_\alpha + A(f_\alpha)} = f_\alpha \circ l_{g_\alpha}^{-1}|_{g_\alpha + A(f_\alpha)}$ it follows that $f_{g_\alpha}(d_i) = f(g_\alpha + d_i) = x_i$. Clearly we can suppose that $\alpha \neq 0$. Thanks to the construction of the family $\{g_\alpha + A(f_\alpha)\}$ in Korovin's orbit we have g_α that is not in the minimal subgroup generated by $A(f_\alpha)$. Thus $g_\alpha, d_1 + g_\alpha, \ldots, d_m + g_\alpha \neq 0$. $\square$

Korovin's orbit with a group G of order 2 is a Hausdorff quasi-topological group that has not a group weak (and semi-) completion:

Example 6. Take Korovin's orbit Y described in Example 5. Let G be a group of order 2 so that Y would be quasi-topological.

Denote by $\mathfrak{f}$ the filter generated by the set consisting of all the sets $F_{c_0,\ldots,c_n} = \{f_g; f_g(c_i) = f(c_i), i = 0, \ldots, n\}$, where $c_0, \ldots, c_n \neq 0$.

It holds: $F_{c_0,\ldots,c_n} \cap F_{d_0,\ldots,d_m} = F_{c_0,\ldots,c_n,d_0,\ldots,d_m} \in \mathfrak{f}$ and f is in $\bigcap \mathfrak{f}$. Thus $\mathfrak{f}$ is a filter on Y.

First we show that $\mathfrak{f}$ is weak Cauchy. Take an arbitrary set $F_{c_0,\ldots,c_n} \in \mathfrak{f}$ and a basic neighbourhood $U_{d_0,\ldots,d_m}^f = \{f_g;\ f_g(d_i) \in U_i,\ i = 0, \ldots, m\}$ of $f_0 = f$, where U_i are neighbourhoods of $f(d_i)$ in X, $d_i \in G$, $i = 0, \ldots, m$.

If $d_i \neq 0$ for all $i = 0, \ldots, m$ then $U_{d_0,\ldots,d_m}^f \supset F_{d_0,\ldots,d_m} \in \mathfrak{f}$. We have $f_0 = f \in F_{c_0,\ldots,c_n}$ and $f_0 + U_{d_0,\ldots,d_m}^f = U_{d_0,\ldots,d_m}^f \in \mathfrak{f}$.

Take now $d_0 = 0$. Thanks to the previous lemma we find such an $\alpha \neq 0$ that $f_{g_\alpha}(c_i) = f(c_i)$, $i = 0, \ldots, n$ (which means that $f_{g_\alpha} \in F_{c_0,\ldots,c_n}$) and $f_{g_\alpha}(d_i) = f(d_i)$, $i = 0, \ldots, m$ and $g_\alpha, d_1 + g_\alpha, \ldots, d_m + g_\alpha \neq 0$. The set $F_{g_\alpha, d_1+g_\alpha,\ldots,d_m+g_\alpha} = \{f_g;\ f_g(d_i + g_\alpha) = f(d_i + g_\alpha) = f(d_i) = f_{g_\alpha}(d_i + g_\alpha),\ i = 0, \ldots, m\} \in \mathfrak{f}$. The last set is included in a neighbourhood $U^{f_{g_\alpha}}_{d_0+g_\alpha,\ldots,d_m+g_\alpha} = U^f_{d_0,\ldots,d_m} + f_{g_\alpha}$ of f_{g_α}, which finally must belong to the filter $\mathfrak{f}$.

Filter $\mathfrak{f}$ is weak Cauchy.

It cannot be Cauchy. In fact the function $f = f_0$ is in $\bigcap \mathfrak{f}$ and if $\mathfrak{f}$ were Cauchy it would converge to f. It does not because thanks to the previous lemma for any point $x \in X \setminus V$, where V is a neighbourhood of $f(0)$ in X, and any $F_{c_0,\ldots,c_n}$, where $c_0, \ldots, c_n \neq 0$, we find an $\alpha < \tau$ such that $f(c_i + g_\alpha) = f(c_i)$, $i = 0, \ldots, n$, and $f(g_\alpha) = x$. Thus the neighbourhood $V^f = \{f_d;\ f_d(0) \in V\}$ of f is not in the filter $\mathfrak{f}$ because for every $F_{c_0,\ldots,c_n}$ from the base of the filter $\mathfrak{f}$ we can find a function $f_{g_\alpha} \in F_{c_0,\ldots,c_n} \setminus V^f$.

From the last example it follows that Korovin's orbits are not topological groups because in topological groups Cauchy, weak Cauchy and semi-Cauchy filters coincide.

There still remains an important question:

Question 1. Do quasi-topological groups have group classic completions or completions?

References

[1] A.V. Archangelskii, M. Hušek, Extensions of topological and semitopological groups and the product operation, Comment. Math. Univ. Carolin. 42 (1) (2001) 173–186.
[2] A.V. Archangelskii, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press, Paris, 2008.
[3] B. Batíková, Completion of semi-uniform spaces, Appl. Categor. Struct. 15 (2007) 483–491.
[4] E. Čech, Topological Spaces, Academia, Prague, 1966.
[5] R. Engelking, General Topology, PWN—Polish Scientific Publishers, Warszawa, 1977.
[6] A.V. Korovin, Continuous actions of pseudocompact groups and axioms of topological groups, Comment. Math. Univ. Carolin. 33 (2) (1992) 335–343.
[7] H.P.A. Künzi, Quasi-uniform spaces in the year 2001, in: M. Hušek, J. van Mill (Eds.), Recent Progress in General Topology, II, North-Holland, Amsterdam, 2002, pp. 313–344.
[8] W. Roelcke, S. Dierolf, Uniform Structures on Topological Groups and their Quotients, McGraw–Hill, New York, 1981.